CONTACT PROBLEMS FOR A HALF-SPACE WITH AN INCLUSION

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In this paper solutions of two axisymmetric contact problems are given. These involve an elastic half-space with a cylindrical hole and a half-space made of two materials separated by the surface $r = a$.

1. The first problem we shall investigate is the torsion of an elastic half-space with a circular cylindrical hole when twisting is accomplished by rotation of a rigid circular die of radius $r = b$ which is rigidly attached to the half-space symmetrically with respect to the hole (Fig.1). On the remaining parts of the surface of the hollow half-space, arbitrary (but absolutely summable) tangential tractions act. As is well known, this problem reduces to the determination of a displacement function $\Phi(r, z)$ which satisfies Michell's equation

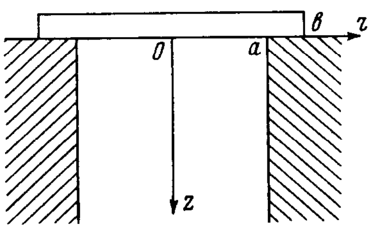

Fig. 1

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{3}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad (1.1)$$

in the region of an axial section of the body of revolution, and which also satisfies the boundary conditions

$$v(r, 0) = f_1(r) = \varkappa r \qquad (1 \leqslant r < b)$$
$$\tau_z(r, 0) = f_2(r) \qquad (b < r < \infty)$$
$$\tau_r(1, z) = f_3(r) \quad (0 \leqslant z < \infty) \qquad (1.2)$$

Here $\varkappa$ is the angle of rotation of the die. The shear stresses τ_r, τ_z and the displacement v are determined in terms of the displacement function $\Phi(r, z)$ by Formulas

$$\tau_r = Gr\frac{\partial\Phi}{\partial r}, \qquad \tau_z = Gr\frac{\partial\Phi}{\partial z} \qquad v = r\Phi(r, z) \qquad (1.3)$$

Following Sneddon [1], we represent the solution in the form of the sum of Fourier-Bessel integrals

$$\Phi(r, z) = \int_0^\infty \frac{\psi(\xi)}{\xi r} e^{-\xi z} J_1(\xi r)\, d\xi + \int_0^\infty \frac{\chi(\xi)}{\xi r} K_1(\xi r) \sin \xi z dz \qquad (1.4)$$

where $J_n(x)$ is a Bessel function of the first kind, of a real argument, $K_n(x)$ is a Bessel function of the second kind, of imaginary argument, and $\psi(\xi)$ and $\chi(\xi)$ are unknown functions.

We express the tangential displacement and shear stresses in terms of these integrals

$$v(r, z) = \int_0^\infty \xi^{-1}\psi(\xi) J_1(\xi r) e^{-\xi z} d\xi + \int_0^\infty \xi^{-1}\chi(\xi) K_1(\xi r) \sin \xi z d\xi$$

$$\tau_z(r, z) = -G\int_0^\infty \psi(\xi) J_1(\xi r) e^{-\xi z} d\xi + G\int_0^\infty \chi(\xi) K_1(\xi r) \cos \xi z\, d\xi \qquad (1.5)$$

$$\tau_r(r, z) = -G\int_0^\infty \psi(\xi) J_2(\xi r) e^{-\xi z} d\xi - G\int_0^\infty \chi(\xi) K_2(\xi r) \sin \xi z d\xi$$

By satisfying the boundary conditions (1.2) we obtain the following system of integral equations for the determination of the unknown functions $\psi(\xi)$ and $\chi(\xi)$:

$$\int_0^\infty \xi^{-1}\psi(\xi) J_1(\xi r)\, d\xi = f_1(r) \qquad (1 \leqslant r < b)$$

$$\int_0^\infty \psi(\xi) J_1(\xi r)\, d\xi = g(r) \qquad (b < r < \infty) \qquad (1.6)$$

$$\int_0^\infty \psi(\xi) J_2(\xi) e^{-\xi z} d\xi + \int_0^\infty \chi(\xi) K_2(\xi) \sin \xi z\, d\xi + \frac{1}{G} f_3(z) = 0 \qquad (0 \leqslant z < \infty) \qquad (1.7)$$

where

$$g(r) = \int_0^\infty \chi(\xi) K_1(\xi r)\, d\xi - \frac{1}{G} f_2(r) \qquad (1.8)$$

Dual integral equations of the type (1.6), where $0 \le r < \infty$ have been considered in many works [2 and 3]. If the region of variation of r is $1 \le r < \infty$, then a new equation of type (1.7) must be appended to Equations (1.6) for completeness. A similar system of equations was investigated in [4] by Srivastav. However, the author considers only the case $f_3(z) = 0$, $f_1(r) = 0$, the last condition playing a key role in the solution of the equations. In the present problem, the case $f_1(r) = 0$ is of no interest because we shall consider $\varkappa \neq 0$.

Using the result of [2 and 3], we seek the solution of the dual integral equations in the form

$$\psi(\xi) = \left(\frac{2\xi^3}{\pi}\right)^{1/2} \left[\int_1^b y^{3/2} F_1(y) J_{1/2}(\xi y)\, dy - \int_b^\infty y^{3/2} F_2(y) J_{1/2}(\xi y)\, dy\right] \qquad (1.9)$$

The function $F_1(y)$ is determined by Formula

$$F_1(y) = \frac{1}{y^2} \frac{d}{dy} \int_1^y \frac{u^2 f_1(u)\, du}{(y^2 - u^2)^{1/2}}$$

The function $F_2(y)$ satisfies the condition

$$\lim_{y\to\infty} yF_2(y) = 0 \qquad (1.10)$$

The expression (1.9) satisfies the first equation of (1.6). In order that it satisfies the second equation of (1.6), the function $F_2(y)$ must have the form

$$F_2(y) + \int_y^\infty \frac{g(r)\,dr}{(r^2-y^2)^{1/2}} = 0 \qquad (1.11)$$

as follows from [2 and 3].

Substituting the value of $g(r)$ from (1.8) into (1.11), we obtain

$$yF_2(y) + \pi\int_0^\infty \xi^{-1}\chi(\xi)\,e^{-\xi y}\,d\xi - \frac{y}{G}\int_y^\infty \frac{f_2(r)\,dr}{(r^2-y^2)^{1/2}} = 0 \qquad (1.12)$$

We now express the function $\chi(\xi)$ in terms of $\psi(\xi)$ by Equation (1.7)

$$\frac{\pi}{2}\chi(\xi)K_2(\xi) + \xi\int_0^\infty \frac{\psi(\alpha)J_2(\alpha)}{\alpha^2+\xi^2}\,d\alpha + \frac{1}{G}\int_0^\infty f_3(z)\sin\xi z\,dz = 0$$

Substituting the value of the function $\psi(\xi)$ from (1.9) into the last equation, we obtain

$$\chi(\xi) + \frac{2\xi I_2(\xi)}{\pi K_2(\xi)}\left[\int_b^\infty yF_2(y)\,e^{-y\xi}\,dy - \int_1^b yF_1(y)\,e^{-y\xi}\,dy\right] + \frac{2}{\pi G K_2(\xi)}\int_0^\infty f_3(z)\sin\xi z\,dz = 0 \qquad (1.13)$$

where $I_n(x)$ is a Bessel function of the first kind, of imaginary argument. To obtain the value of the integral

$$\int_0^\infty x^{3/2}J_2(ax)J_{1/2}(yx)\frac{dx}{x^2+\xi^2} = -\left(\frac{\pi}{2y}\right)I_2(a\xi)\,e^{-y\xi} \qquad (y>a)$$

has been used.

Eliminating the function $\chi(\xi)$ from Equations (1.12) and (1.13), we obtain a Fredholm integral equation of the second kind for the determination of the function $f(y) = yF_2(y)$

$$f(y) + \int_b^\infty K(x+y)f(x)\,dx = F(y) \qquad (b<y<\infty) \qquad (1.14)$$

In this expression, the following notation has been introduced:

$$K(z) = -2\int_0^\infty \frac{I_2(\xi)}{K_2(\xi)}e^{-\xi z}\,d\xi \qquad (z \geqslant 2b > 2) \qquad (1.15)$$

$$F(y) = \frac{y}{G}\int_y^\infty \frac{f_2(r)\,dr}{(r^2-y^2)^{1/2}} + \frac{2}{G}\int_0^\infty \frac{e^{-\xi y}d\xi}{\xi K_2(\xi)}\int_0^\infty f_3(z)\sin\xi z\,dz - \int_1^b rF_1(r)K(r+y)\,dr \qquad (1.16)$$

where $I_2(x)$ and $K_2(x)$ are Bessel functions of the first and second kinds, of imaginary argument.

It follows from Equations (1.14) to (1.16) that the condition (1.10) on the function $F_2(y)$ is equivalent to the condition

$$\lim_{y\to\infty} y \int_y^\infty \frac{f_2(r)\,dr}{(r^2-y^2)^{1/2}} = 0 \tag{1.17}$$

This condition will hold if $f_2(r)$ tends to zero like $r^{-\alpha}$ as $r\to\infty$, with $\alpha > 1+\epsilon$. However, from the condition of absolute summability of the external shearing tractions it follows that $f_2(r) = O(r^{-2-\epsilon_1})$, with $\epsilon_1 > 0$. Therefore, the condition (1.17) is satisfied. The right-hand side of the integral equation, $F(y)$, then goes to zero like $y^{-2-\epsilon_1}$ as $y\to\infty$. If $f_2(r) = 0$ for $b < c < r$ then the function $F(y)$ approaches zero exponentially. The kernel of the integral equation, (1.15), is a continuous and monotonically decreasing function for $b > 1$.

We shall now show that the integral equation (1.14) can be solved by the method of successive approximations. In order to do this, the integrals

$$u(b) = \int_b^\infty \int_b^\infty K^2(x+y)\,dx\,dy = \int_0^\infty x K^2(x+2b)\,dx$$

$$-v(b) = \int_b^\infty K(t+b)\,dt = \int_0^\infty K(x+2b)\,dx$$

must be computed.

We give some values of these integrals which were computed on the "Nairi" digital computer

$b =$	1.05	1.10	1.20	1.50	2.00
$u(b) =$	0.1368	0.0396	0.0029	0.00003	—
$-v(b) =$	0.4421	0.2234	0.0982	0.0269	0.0050

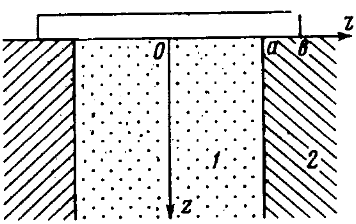

Fig. 2

As is apparent from the results, the integral equation (1.14) can be successfully solved by the method of successive approximations for $b \geq 1.05$. The larger the value of b, the more rapid is the convergence of this process.

The question of whether the integral equation (1.14) is solvable for $1 < b < 1.05$ remains unsettled. For $b = 1$, we have $u(1) = -v(1) = \infty$. But in this case, as may be easily observed, the problem which has been formulated can be solved exactly without using integral equations. This solution can be obtained by elementary means if in the first integral of Equation (1.4) the function $J_1(\xi r)$ is replaced by the function

$$W_1(\xi, r) = J_1(\xi r)\,Y_2(\xi a) - J_2(\xi a)\,Y_1(\xi r)$$

where $Y_n(x)$ is a Bessel function of the second kind, of a real argument.

2. We shall now consider the second problem of torsion of a half-space consisting of two materials when the surface separating the materials is the cylindrical surface $r = a$. The half-space is twisted by rotating a rigid circular die of radius b $(b > a)$ which is rigidly attached to both materials and is symmetrically located with respect to the inclusion (Fig.2).

The boundary conditions for this problem have the form

$$v(r,0) = f_1(r) \quad (0 \leqslant r < b), \qquad \tau_z(r,0) = f_2(r) \quad (b < r < \infty) \tag{2.1}$$

We seek a displacement function $\Phi(r, z)$ in the form

$$\Phi(r, z) = \begin{cases} \Phi_1(r, z) & (0 \leqslant r \leqslant a,\ 0 \leqslant z < \infty) \\ \Phi_2(r, z) & (a < r < \infty,\ 0 \leqslant z < \infty) \end{cases} \tag{2.2}$$

It follows from (2.1) and (2.2) that the functions $\Phi_i(r, z)$ $(i = 1, 2)$ satisfy the differential equation (1.1), the boundary conditions

$$v_1(r, 0) = f_1(r) \quad (0 \leqslant r \leqslant a), \qquad v_2(r, 0) = f_1(r) \quad (a \leqslant r < b)$$
$$\tau_z^{(2)}(r, 0) = f_2(r) \qquad (b < r < \infty) \tag{2.3}$$

and the conditions which match the solutions in the interior and exterior parts of the half-space

$$\Phi_1(a, z) = \Phi_2(a, z), \qquad G_1 \frac{\partial \Phi_1}{\partial r}\bigg|_{r=a} = G_2 \frac{\partial \Phi_2}{\partial r}\bigg|_{r=a} \tag{2.4}$$

Here the stresses $\tau_r^{(i)}$, $\tau_z^{(i)}$ and the displacement v_i are determined in terms of the displacement function $\Phi(r, z)$ by Equations (1.3), where G and Φ appear with the subscript i , $(i = 1, 2)$.

We represent the functions Φ_1 and Φ_2 in the following form:

$$\Phi_1(r, z) = \int_0^\infty C(\xi) \frac{I_1(\xi r)}{\xi r} \sin \xi z \, d\xi + \sum_{k=1}^\infty A_k e^{-\mu_k z} \frac{J_1(\mu_k r)}{\mu_k r}$$
$$\Phi_2(r, z) = \int_0^\infty D(\xi) \frac{K_1(\xi r)}{\xi r} \sin \xi z d\xi + \int_0^\infty B(\xi) e^{-\xi z} \frac{J_1(\xi r)}{\xi r} d\xi \tag{2.5}$$

where μ_k are the positive roots of $J_1(\mu a) = 0$.

The value of A_k is obtained from the first condition of (2.3)

$$A_k = \frac{2\mu_k}{a^2 J_0^2(\mu_k a)} \int_0^a r f_1(r) J_1(\mu_k r)\, dr = -\frac{2\varkappa}{J_0(\mu_k a)} \tag{2.6}$$

The last two conditions of (2.3) provide the dual integral equations for the determination of the function $B(\xi)$

$$\int_0^\infty \xi^{-1} B(\xi) J_1(\xi r)\, d\xi = f_1(r) \quad (a \leqslant r \leqslant b)$$
$$\int_0^\infty B(\xi) J_1(\xi r)\, d\xi = g(r) \qquad (b < r < \infty) \tag{2.7}$$

the right-hand sides of which depend on the unknown function $D(\xi)$

$$g(r) = \int_0^\infty D(\xi) K_1(\xi r)\, d\xi - \frac{1}{G_2} f_2(r) \tag{2.8}$$

The following system of equations is obtained from the conditions (2.4) to find the unknown functions $C(\xi)$ and $D(\xi)$:

$$C(\xi) I_1(\xi a) - D(\xi) K_1(\xi a) = \frac{2\xi^2}{\pi} \int_0^\infty \frac{B(\alpha) J_1(\alpha a)}{\alpha(\alpha^2 + \xi^2)} d\alpha \tag{2.9}$$
$$G_1 C(\xi) I_2(\xi a) + G_2 D(\xi) K_2(\xi a) = \frac{2\xi}{\pi}\left[G_1 \frac{\varkappa a}{\xi} \frac{I_2(\xi a)}{I_1(\xi a)} - G_2 \int_0^\infty \frac{B(\alpha) J_2(\alpha a)}{\alpha^2 + \xi^2} d\alpha\right]$$

From the system (2.9), we have (2.10)

$$C(\xi)=\frac{2\xi}{\pi\Delta}\left\{G_1\frac{\varkappa a}{\xi}\frac{K_1(\xi a)I_2(\xi a)}{I_1(\xi a)}+G_2\int_0^\infty\frac{\xi K_2(\xi a)J_1(\alpha a)-\alpha K_1(\xi a)J_2(\alpha a)}{\alpha(\alpha^2+\xi^2)}B(\alpha)\,d\alpha\right\}$$

$$D(\xi)=\frac{2\xi}{\pi\Delta}\left\{G_1\frac{\varkappa a}{\xi}I_2(\xi a)-\int_0^\infty\frac{G_1\xi I_2(\xi a)J_2(\alpha a)+G_2\alpha I_1(\xi a)J_2(\alpha a)}{\alpha(\alpha^2+\xi^2)}B(\alpha)\,d\alpha\right\}$$

where

$$\Delta=G_1K_1(\xi a)I_2(\xi a)+G_2K_2(\xi a)I_1(\xi a) \qquad (2.11)$$

We now represent the solution of the dual equations (2.7) in the form

$$B(\xi)=\left(\frac{2\xi^3}{\pi}\right)^{1/2}\left[\int_a^b y^{3/2}F_1(y)J_{1/2}(\xi y)\,dy-\int_b^\infty y^{3/2}F_2(y)J_{1/2}(\xi y)\,dy\right] \qquad (2.12)$$

$$F_1(y)=\frac{1}{y^2}\frac{d}{dy}\int_a^y\frac{u^2f_1(u)\,du}{(r^2-y^2)^{1/2}},\quad F_2(y)+\int_y^\infty\frac{g(r)\,dr}{(r^2-y^2)^{1/2}}=0,\quad \lim_{y\to\infty}yF_2(y)=0$$

Substituting the value of $g(r)$ from (2.8) into (2.12), we obtain

$$yF_2(y)+\pi\int_0^\infty\xi^{-1}D(\xi)e^{-\xi y}\,d\xi-\frac{y}{G_2}\int_y^\infty\frac{f_2(r)\,dr}{(r^2-y^2)^{1/2}}=0 \qquad (2.13)$$

The condition of (2.13) in the function $F_2(y)$ now assumes the form

$$\lim_{y\to\infty}y\int_y^\infty\frac{f_2(r)\,dr}{(r^2-y^2)^{1/2}}=0 \qquad (2.14)$$

However, since it follows from the condition of absolute summability of the external shearing tractions that $f_2(r)=O(r^{-2-\varepsilon})$, the condition (2.14) is satisfied. The condition of (2.12) is then also satisfied.

Let us express $D(\xi)$ and $C(\xi)$ in terms of the function $F_2(y)$. To do this we substitute the value of $B(\xi)$ from (2.12) into (2.14)

$$D(\xi)=\frac{2\xi(G_1-G_2)}{\pi\Delta}I_1(\xi a)I_2(\xi a)\left[\frac{G_1\varkappa aI_2(\xi a)}{(G_1-G_2)\xi I_1^2(\xi a)}-\right.$$
$$\left.-\int_a^b yF_1(y)e^{-\xi y}\,dy+\int_b^\infty yF_2(y)e^{-\xi y}\,dy\right] \qquad (2.15)$$

$$C(\xi)=\frac{2G_2}{\pi a\Delta}\left[\frac{G_1\varkappa a^2I_2(\xi a)K_1(\xi a)}{G_2I_1(\xi a)}+\int_a^b yF_1(y)e^{-\xi y}\,dy-\int_b^\infty yF_2(y)e^{-\xi y}dy\right]$$

Eliminating the function $D(\xi)$ from Equations (2.13) and (2.15), we obtain the following Fredholm integral equation of the second kind for the determination of the unknown function $F_2(y)$:

$$f(y)+\int_b^\infty f(x)K(x+y)\,dx=F(y)\qquad (b\leqslant y<\infty) \qquad (2.16)$$

where

$$f(y) = yF_2(y), \qquad K(z) = 2(G_1 - G_2)\int_0^\infty \frac{I_1(\xi a)\, I_2(\xi a)}{\Delta(\xi a)} e^{-\xi z}\, d\xi$$

$$F(y) = \int_a^b xF_1(x) K(x+y)\, dx + \frac{y}{G_2}\int_y^\infty \frac{f_2(r)\, dr}{(r^2 - y^2)^{1/2}} - 2G_1 \varkappa a \int_0^\infty \frac{I_2(\xi a)\, e^{-\xi y}}{\xi \Delta(\xi a)}\, d\xi$$

The possibility of application of the method of successive approximations to the solution of the integral equation (2.16) can be demonstrated by a method analogous to the one used in Section 1.

As a special case we obtain for $G_1 = 0$ the solution of the problem considered in Section 1. In the case $G_1 = G_2$, the kernel of the integral equation reduces to zero and we obtain the exact solution of the problem of torsion of a homogeneous elastic half-space which was considered by Rostovtsev [5]. In order to obtain the solution in the form of [5] it is also necessary to set $a = 0$.

By setting $G_1 = \infty$ we obtain the solution of the problem of torsion of an elastic half-space with a rigid inclusion when twisting is accomplished by rotation of a rigid die of radius b rigidly attached to the half-space and symmetrically located with respect to the inclusion. For this case, the kernel of the integral equation assumes the form

$$K(z) = 2\int_0^\infty \frac{I_1(\xi a)\, e^{-\xi z}}{K_1(\xi a)}\, d\xi \qquad (z \geqslant 2b > 2a) \tag{2.17}$$

and the right-hand side takes the form

$$F(y) = \int_a^b xF_1(x) K(x+y)\, dx + \frac{y}{G_2}\int_y^\infty \frac{f_2(r)\, dr}{(r^2 - y^2)^{1/2}} - 2\varkappa a \int_0^\infty \frac{e^{-\xi y} d\xi}{\xi K_1(\xi a)} \tag{2.18}$$

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